

# Structural Refinement for the Modal $\mu$ -Calculus

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**Abstract.** We introduce a new notion of structural refinement, a sound abstraction of logical implication, for the modal  $\mu$ -calculus. Using new translations between the modal  $\mu$ -calculus and disjunctive modal transition systems, we show that these two specification formalisms are structurally equivalent.

Using our translations, we also transfer the structural operations of composition and quotient from disjunctive modal transition systems to the modal  $\mu$ -calculus. This shows that the modal  $\mu$ -calculus supports composition and decomposition of specifications.

## 1 Introduction

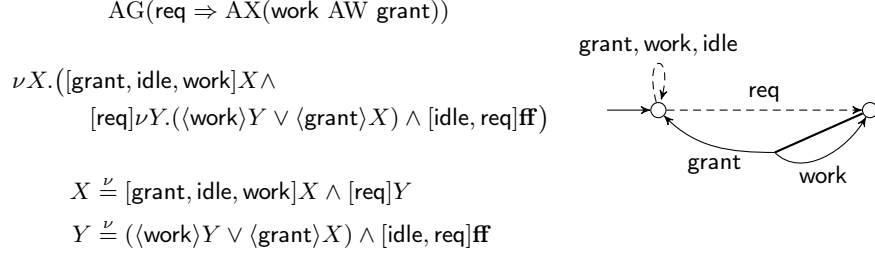
There are two conceptually different approaches for the specification and verification of properties of formal models. *Logical* approaches make use of logical formulae for expressing properties and then rely on efficient model checking algorithms for verifying whether or not a model satisfies a formula. *Automata*-based approaches, on the other hand, exploit equivalence or refinement checking for verifying properties, given that models and properties are specified using the same (or a closely related) formalism.

The logical approaches have been quite successful, with a plethora of logical formalisms available and a number of successful model checking tools. One particularly interesting such formalism is the modal  $\mu$ -calculus [21], which is universal in the sense that it generalizes most other temporal logics, yet mathematically simple and amenable to analysis.

One central problem in the verification of formal properties is *state space explosion*: when a model is composed of many components, the state space of the combined system quickly grows too big to be analyzed. To combat this problem, one approach is to employ *compositionality*. When a model consists of several components, each component would be model checked by itself, and then the components' properties would be composed to yield a property which automatically is satisfied by the combined model.

Similarly, given a global property of a model and a component of the model that is already known to satisfy a local property, one would be able to *decompose* automatically, from the global property and the local property, a new property which the rest of the model must satisfy. We refer to [23] for a good account of these and other features which one would wish specifications to have.

As an alternative to logical specification formalisms and with an eye to compositionality and decomposition, automata-based *behavioral* specifications were



**Fig. 1.** An example property specified in CTL (top left), in the modal  $\mu$ -calculus (below left), as a modal equation system (third left), and as a DMTS (right).

introduced in [22]. Here the specification formalism is a generalization of the modeling formalism, and the satisfaction relation between models and specifications is generalized to a refinement relation between specifications, which resembles simulation and bisimulation and can be checked with similar algorithms.

For an example, we refer to Fig. 1 which shows the property informally specified as “after a *req*(uest), no *idle*(ing) is allowed, but only *work*, until *grant* is executed” using the logical formalisms of CTL [15] and the modal  $\mu$ -calculus [21] and the behavioral formalism of disjunctive modal transition systems [26].

The precise relationship between logical and behavioral specification formalisms has been subject to some investigation. In [22], Larsen shows that any *modal transition system* can be translated to a formula in Hennessy-Milner logic which is equivalent in the sense of admitting the same models. Conversely, Boudol and Larsen show in [11] that any formula in Hennessy-Milner logic is equivalent to a finite disjunction of modal transition systems.

We have picked up this work in [6], where we show that any *disjunctive modal transition system* (DMTS) is equivalent to a formula in the *modal  $\nu$ -calculus*, the safety fragment of the modal  $\mu$ -calculus which uses only maximal fixed points, and vice versa. (Note that the modal  $\nu$ -calculus is equivalent to Hennessy-Milner logic with recursion and maximal fixed points.) Moreover, we show in [6] that DMTS are as expressive as (non-deterministic) *acceptance automata* [30, 31]. Together with the inclusions of [7], this settles the expressivity question for behavioral specifications: they are at most as expressive as the modal  $\nu$ -calculus.

In this paper, we show that not only are DMTS as expressive as the modal  $\nu$ -calculus, but the two formalisms are *structurally equivalent*. Introducing a new notion of structural refinement for the modal  $\nu$ -calculus (a sound abstraction of logical implication), we show that one can freely translate between the modal  $\nu$ -calculus and DMTS, while preserving structural refinement.

DMTS form a *complete specification theory* [2] in that they both admit logical operations of conjunction and disjunction and structural operations of composition and quotient [6]. Hence they support full compositionality and decomposition in the sense of [23]. Using our translations, we can transport these notions

to the modal  $\nu$ -calculus, thus also turning the modal  $\nu$ -calculus into a complete specification theory.

In order to arrive at our translations, we first recall DMTS and (non-deterministic) *acceptance automata* in Section 2. We also introduce a new hybrid modal logic, which can serve as compact representation for acceptance automata and should be of interest in itself. Afterwards we show, using the translations introduced in [6], that these formalisms are structurally equivalent.

In Section 3 we recall the modal  $\nu$ -calculus and review the translations between DMTS and the modal  $\nu$ -calculus which were introduced in [6]. These in turn are based on work by Boudol and Larsen in [11, 22], hence fairly standard. We show that, though semantically correct, the two translations are structurally *mismatched* in that they relate DMTS refinement to two different notions of  $\nu$ -calculus refinement. To fix the mismatch, we introduce a new translation from the modal  $\nu$ -calculus to DMTS and show that using this translation, the two formalisms are structurally equivalent.

In Section 4, we use our translations to turn the modal  $\nu$ -calculus into a complete specification theory. We remark that all our translations and constructions are based on a new *normal form* for  $\nu$ -calculus expressions, and that turning a  $\nu$ -calculus expression into normal form may incur an exponential blow-up. However, the translations and constructions preserve the normal form, so that this translation only need be applied once in the beginning.

We also note that composition and quotient operators are used in other logics such as *e.g.* spatial [14] or separation logics [32, 28]. However, in these logics they are treated as *first-class* operators, *i.e.* as part of the formal syntax. In our approach, on the other hand, they are defined as operations on logical expressions which as results again yield logical expressions (without compositions or quotients).

Note that some proofs have been relegated to a separate appendix.

## 2 Structural Specification Formalisms

Let  $\Sigma$  be a finite set of labels. A *labeled transition system* (LTS) is a structure  $\mathcal{I} = (S, S^0, \longrightarrow)$  consisting of a finite set of *states*  $S$ , a subset  $S^0 \subseteq S$  of *initial states* and a *transition relation*  $\longrightarrow \subseteq S \times \Sigma \times S$ .

**Disjunctive modal transition systems.** A *disjunctive modal transition system* (DMTS) is a structure  $\mathcal{D} = (S, S^0, \dashrightarrow, \longrightarrow)$  consisting of finite sets  $S \supseteq S^0$  of states and initial states, a *may-transition* relation  $\dashrightarrow \subseteq S \times \Sigma \times S$ , and a *disjunctive must-transition* relation  $\longrightarrow \subseteq S \times 2^{\Sigma \times S}$ . It is assumed that for all  $(s, N) \in \longrightarrow$  and all  $(a, t) \in N$ ,  $(s, a, t) \in \dashrightarrow$ .

As customary, we write  $s \xrightarrow{a} t$  instead of  $(s, a, t) \in \dashrightarrow$ ,  $s \longrightarrow N$  instead of  $(s, N) \in \longrightarrow$ ,  $s \xrightarrow{a}$  if there exists  $t$  for which  $s \dashrightarrow t$ , and  $s \not\xrightarrow{a}$  if there does not.

The intuition is that may-transitions  $s \dashrightarrow t$  specify which transitions are permitted in an implementation, whereas a must-transitions  $s \longrightarrow N$  stipulates a disjunctive requirement: at least one of the choices  $(a, t) \in N$  must be imple-

mented. A DMTS  $(S, S^0, \dashrightarrow, \longrightarrow)$  is an *implementation* if  $\longrightarrow = \{(s, \{(a, t)\}) \mid s \xrightarrow{a} t\}$ ; DMTS implementations are precisely LTS.

DMTS were introduced in [26] in the context of equation solving, or *quotient*, for specifications and are used *e.g.* in [5] for LTL model checking. They are a natural closure of *modal transition systems* (MTS) [22] in which all disjunctive must-transitions  $s \longrightarrow N$  lead to singletons  $N = \{(a, t)\}$ .

Let  $\mathcal{D}_1 = (S_1, S_1^0, \dashrightarrow_1, \longrightarrow_1)$ ,  $\mathcal{D}_2 = (S_2, S_2^0, \dashrightarrow_2, \longrightarrow_2)$  be DMTS. A relation  $R \subseteq S_1 \times S_2$  is a *modal refinement* if it holds for all  $(s_1, s_2) \in R$  that

- for all  $s_1 \xrightarrow{a} t_1$  there is  $t_2 \in S_2$  with  $s_2 \xrightarrow{a} t_2$  and  $(t_1, t_2) \in R$ , and
- for all  $s_2 \longrightarrow N_2$  there is  $s_1 \longrightarrow N_1$  such that for each  $(a, t_1) \in N_1$  there is  $(a, t_2) \in N_2$  with  $(t_1, t_2) \in R$ .

We say that  $\mathcal{D}_1$  *modally refines*  $\mathcal{D}_2$ , denoted  $\mathcal{D}_1 \leq_m \mathcal{D}_2$ , whenever there exists a modal refinement  $R$  such that for all  $s_1^0 \in S_1^0$ , there exists  $s_2^0 \in S_2^0$  for which  $(s_1^0, s_2^0) \in R$ . We write  $\mathcal{D}_1 \equiv_m \mathcal{D}_2$  if  $\mathcal{D}_1 \leq_m \mathcal{D}_2$  and  $\mathcal{D}_2 \leq_m \mathcal{D}_1$ . For states  $s_1 \in S_1$ ,  $s_2 \in S_2$ , we write  $s_1 \leq_m s_2$  if the DMTS  $(S_1, \{s_1\}, \dashrightarrow_1, \longrightarrow_1) \leq_m (S_2, \{s_2\}, \dashrightarrow_2, \longrightarrow_2)$ .

Note that modal refinement is reflexive and transitive, *i.e.* a preorder on DMTS. Also, the relation on states  $\leq_m \subseteq S_1 \times S_2$  defined above is itself a modal refinement, indeed the maximal modal refinement under the subset ordering.

The *set of implementations* of an DMTS  $\mathcal{D}$  is  $\llbracket \mathcal{D} \rrbracket = \{\mathcal{I} \leq_m \mathcal{D} \mid \mathcal{I} \text{ implementation}\}$ . This is, thus, the set of all LTS which satisfy the specification given by the DMTS  $\mathcal{D}$ . We say that  $\mathcal{D}_1$  *thoroughly refines*  $\mathcal{D}_2$ , and write  $\mathcal{D}_1 \leq_{th} \mathcal{D}_2$ , if  $\llbracket \mathcal{D}_1 \rrbracket \subseteq \llbracket \mathcal{D}_2 \rrbracket$ . We write  $\mathcal{D}_1 \equiv_{th} \mathcal{D}_2$  if  $\mathcal{D}_1 \leq_{th} \mathcal{D}_2$  and  $\mathcal{D}_2 \leq_{th} \mathcal{D}_1$ . For states  $s_1 \in S_1$ ,  $s_2 \in S_2$ , we write  $\llbracket s_1 \rrbracket = \llbracket (S_1, \{s_1\}, \dashrightarrow_1, \longrightarrow_1) \rrbracket$  and  $s_1 \leq_{th} s_2$  if  $\llbracket s_1 \rrbracket \subseteq \llbracket s_2 \rrbracket$ .

The below proposition, which follows directly from transitivity of modal refinement, shows that modal refinement is *sound* with respect to thorough refinement; in the context of specification theories, this is what one would expect, and we only include it for completeness of presentation. It can be shown that modal refinement is also *complete* for *deterministic* DMTS [8], but we will not need this here.

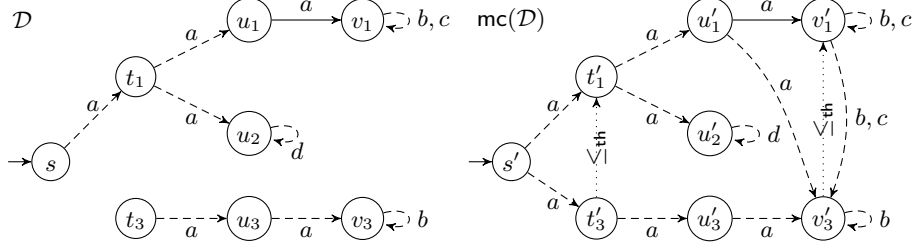
**Proposition 1.** *For all DMTS  $\mathcal{D}_1, \mathcal{D}_2$ ,  $\mathcal{D}_1 \leq_m \mathcal{D}_2$  implies  $\mathcal{D}_1 \leq_{th} \mathcal{D}_2$ .  $\square$*

We introduce a new construction on DMTS which will be of interest for us; intuitively, it adds all possible may-transitions without changing the implementation semantics. The *may-completion* of a DMTS  $\mathcal{D} = (S, S^0, \dashrightarrow, \longrightarrow)$  is  $mc(\mathcal{D}) = (S, S^0, \dashrightarrow_{mc}, \longrightarrow)$  with

$$\dashrightarrow_{mc} = \{(s, a, t') \subseteq S \times \Sigma \times S \mid \exists (s, a, t) \in \dashrightarrow : t' \leq_{th} t\}.$$

Note that to compute the may-completion of a DMTS, one has to decide thorough refinements, hence this computation (or, more precisely, deciding whether a given DMTS is may-complete) is EXPTIME-complete [9]. We show an example of a may-completion in Fig. 2.

**Proposition 2.** *For any DMTS  $\mathcal{D}$ ,  $\mathcal{D} \leq_m mc(\mathcal{D})$  and  $\mathcal{D} \equiv_{th} mc(\mathcal{D})$ .*



**Fig. 2.** A MTS  $\mathcal{D}$  (left) and its may-completion  $\text{mc}(\mathcal{D})$  (right). In  $\text{mc}(\mathcal{D})$ , the semantic inclusions which lead to extra may-transitions are depicted with dotted arrows.

*Proof.* It is always the case that  $\mathcal{D} \leq_m \mathcal{D}$ , and adding may transitions on the right side preserves modal refinement. Therefore it is immediate that  $\mathcal{D} \leq_m \text{mc}(\mathcal{D})$ , hence also  $\mathcal{D} \leq_{\text{th}} \text{mc}(\mathcal{D})$ .

To prove that  $\text{mc}(\mathcal{D}) \leq_{\text{th}} \mathcal{D}$ , we consider an implementation  $\mathcal{I} \leq_m \text{mc}(\mathcal{D})$ ; we must prove that  $\mathcal{I} \leq_m \mathcal{D}$ . Write  $\mathcal{D} = (S, S^0, \dashrightarrow, \longrightarrow)$ ,  $\mathcal{I} = (I, I^0, \dashrightarrow_I, \longrightarrow_I)$  and  $\text{mc}(\mathcal{D}) = (S, S^0, \dashrightarrow_{\text{mc}}, \longrightarrow)$ . Let  $R \subseteq I \times S$  be the largest modal refinement between  $\mathcal{I}$  and  $\text{mc}(\mathcal{D})$ . We now prove that  $R$  is also a modal refinement between  $\mathcal{I}$  and  $\mathcal{D}$ . For all  $(i, d) \in R$ :

- For all  $i \dashrightarrow_I i'$ , there exists  $d' \in S$  such that  $d \dashrightarrow_{\text{mc}} d'$  and  $(i', d') \in R$ . Then by definition of  $\dashrightarrow_{\text{mc}}$ , there exists  $d'' \in S$  such that  $d \dashrightarrow d''$  and  $\llbracket d' \rrbracket \subseteq \llbracket d'' \rrbracket$ .  $(i', d') \in R$  implies  $i' \in \llbracket d' \rrbracket$ , which implies  $i' \in \llbracket d'' \rrbracket$ . This means that  $i' \leq_m d''$ , and since  $R$  is the largest refinement relation in  $I \times S$  it must be the case that  $(i', d'') \in R$ .
- The case of must transitions follows immediately, since must transitions are exactly the same in  $\mathcal{D}$  and  $\text{mc}(\mathcal{D})$ .  $\square$

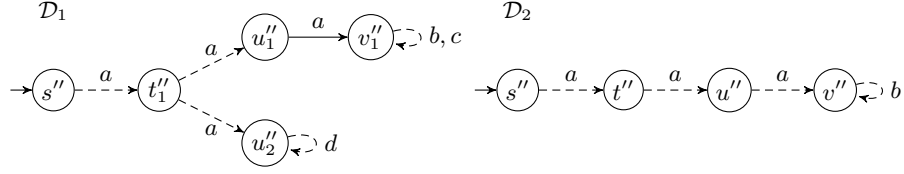
*Example 3.* The example in Fig. 2 shows that generally,  $\text{mc}(\mathcal{D}) \not\leq_m \mathcal{D}$ . First,  $t_3 \leq_{\text{th}} t_1$ : For an implementation  $\mathcal{I} = (I, I^0, \longrightarrow) \in \llbracket t_3 \rrbracket$  with modal refinement  $R \subseteq I \times \{t_3, u_3, v_3\}$ , define  $R' \subseteq I \times \{t_1, u_1, u_2, v_1\}$  by

$$\begin{aligned} R' = \{ & (i, t_1) \mid (i, t_3) \in R \} \cup \{ (i, v_1) \mid (i, v_3) \in R \} \\ & \cup \{ (i, u_1) \mid (i, u_3) \in R, i \xrightarrow{a} \} \\ & \cup \{ (i, u_2) \mid (i, u_3) \in R, i \not\xrightarrow{a} \}, \end{aligned}$$

then  $R'$  is a modal refinement  $\mathcal{I} \leq_m t_1$ . Similarly,  $t'_3 \leq_{\text{th}} t'_1$  in  $\text{mc}(\mathcal{D})$ .

On the other hand,  $t_3 \not\leq_m t_1$  (and similarly,  $t'_3 \not\leq_m t'_1$ ), because neither  $u_3 \leq_m u_1$  nor  $u_3 \leq_m u_2$ . Now in the modal refinement game between  $\text{mc}(\mathcal{D})$  and  $\mathcal{D}$ , the may-transition  $s' \dashrightarrow_I t'_3$  has to be matched by  $s \dashrightarrow_I t_1$ , but then  $t'_3 \not\leq_m t_1$ , hence  $\text{mc}(\mathcal{D}) \not\leq_m \mathcal{D}$ .

Also, the may-completion does not necessarily preserve modal refinement: Consider the DMTS  $\mathcal{D}$  from Fig. 2 and  $\mathcal{D}_1$  from Fig. 3, and note first that  $\text{mc}(\mathcal{D}_1) = \mathcal{D}_1$ . It is easy to see that  $\mathcal{D} \leq_m \mathcal{D}_1$  (just match states in  $\mathcal{D}$  with their

Fig. 3. DMTS  $\mathcal{D}_1$ ,  $\mathcal{D}_2$  from Example 3.

double-prime cousins in  $\mathcal{D}_1$ ), but  $\text{mc}(\mathcal{D}) \not\leq_m \text{mc}(\mathcal{D}_1) = \mathcal{D}_1$ : the may-transition  $s' \xrightarrow{a} t'_3$  has to be matched by  $s'' \xrightarrow{a} t''_1$  and  $t'_3 \not\leq_m t''_1$ .

Lastly, the may-completion can also create modal refinement: Considering the DMTS  $\mathcal{D}_2$  from Fig. 3, we see that  $\mathcal{D}_2 \not\leq_m \mathcal{D}$ , but  $\text{mc}(\mathcal{D}_2) = \mathcal{D}_2 \leq_m \text{mc}(\mathcal{D})$ .

**Acceptance automata.** A (non-deterministic) *acceptance automaton* (AA) is a structure  $\mathcal{A} = (S, S^0, \text{Tran})$ , with  $S \supseteq S^0$  finite sets of states and initial states and  $\text{Tran} : S \rightarrow 2^{2^{\Sigma \times S}}$  an assignment of *transition constraints*. We assume that for all  $s^0 \in S^0$ ,  $\text{Tran}(s^0) \neq \emptyset$ .

An AA is an *implementation* if it holds for all  $s \in S$  that  $\text{Tran}(s) = \{M\}$  is a singleton; hence also AA implementations are precisely LTS. Acceptance automata were first introduced in [30] (see also [31], where a slightly different language-based approach is taken), based on the notion of acceptance trees in [20]; however, there they are restricted to be *deterministic*. We employ no such restriction here. The following notion of modal refinement for AA was also introduced in [30].

Let  $\mathcal{A}_1 = (S_1, S_1^0, \text{Tran}_1)$  and  $\mathcal{A}_2 = (S_2, S_2^0, \text{Tran}_2)$  be AA. A relation  $R \subseteq S_1 \times S_2$  is a *modal refinement* if it holds for all  $(s_1, s_2) \in R$  and all  $M_1 \in \text{Tran}_1(s_1)$  that there exists  $M_2 \in \text{Tran}_2(s_2)$  such that

- $\forall (a, t_1) \in M_1 : \exists (a, t_2) \in M_2 : (t_1, t_2) \in R$ ,
- $\forall (a, t_2) \in M_2 : \exists (a, t_1) \in M_1 : (t_1, t_2) \in R$ .

As for DMTS, we write  $\mathcal{A}_1 \leq_m \mathcal{A}_2$  whenever there exists a modal refinement  $R$  such that for all  $s_1^0 \in S_1^0$ , there exists  $s_2^0 \in S_2^0$  for which  $(s_1^0, s_2^0) \in R$ . Sets of implementations and thorough refinement are defined as for DMTS. Note that as both AA and DMTS implementations are LTS, it makes sense to use thorough refinement and equivalence *across* formalisms, writing *e.g.*  $\mathcal{A} \equiv_{\text{th}} \mathcal{D}$  for an AA  $\mathcal{A}$  and a DMTS  $\mathcal{D}$ .

**Hybrid modal logic.** We introduce a hybrid modal logic which can serve as compact representation of AA. This logic is closely related to the Boolean modal transition systems of [7] and hybrid in the sense of [29, 10]: it contains nominals, and the semantics of a nominal is given as all sets which contain the nominal.

For a finite set  $X$  of nominals, let  $\mathcal{L}(X)$  be the set of formulae generated by the abstract syntax  $\mathcal{L}(X) \ni \phi := \mathbf{tt} \mid \mathbf{ff} \mid \langle a \rangle x \mid \neg \phi \mid \phi \wedge \phi$ , for  $a \in \Sigma$  and  $x \in X$ . The semantics of a formula is a set of subsets of  $\Sigma \times X$ , given as follows:  $\llbracket \mathbf{tt} \rrbracket = 2^{\Sigma \times X}$ ,  $\llbracket \mathbf{ff} \rrbracket = \emptyset$ ,  $\llbracket \neg \phi \rrbracket = 2^{\Sigma \times X} \setminus \llbracket \phi \rrbracket$ ,  $\llbracket \langle a \rangle x \rrbracket = \{M \subseteq \Sigma \times X \mid (a, x) \in M\}$ , and  $\llbracket \phi \wedge \psi \rrbracket = \llbracket \phi \rrbracket \cap \llbracket \psi \rrbracket$ . We also define disjunction  $\phi_1 \vee \phi_2 = \neg(\phi_1 \wedge \phi_2)$ .

An  $\mathcal{L}$ -expression is a structure  $\mathcal{E} = (X, X^0, \Phi)$  consisting of finite sets  $X^0 \subseteq X$  of variables and a mapping  $\Phi : X \rightarrow \mathcal{L}(X)$ . Such an expression is an *implementation* if  $\llbracket \Phi(x) \rrbracket = \{M\}$  is a singleton for each  $x \in X$ . It can easily be shown that  $\mathcal{L}$ -implementations precisely correspond to LTS.

Let  $\mathcal{E}_1 = (X_1, X_1^0, \Phi_1)$  and  $\mathcal{E}_2 = (X_2, X_2^0, \Phi_2)$  be  $\mathcal{L}$ -expressions. A relation  $R \subseteq X_1 \times X_2$  is a *modal refinement* if it holds for all  $(x_1, x_2) \in R$  and all  $M_1 \in \llbracket \Phi_1(x_1) \rrbracket$  that there exists  $M_2 \in \llbracket \Phi_2(x_2) \rrbracket$  such that

- $\forall (a, t_1) \in M_1 : \exists (a, t_2) \in M_2 : (t_1, t_2) \in R$ ,
- $\forall (a, t_2) \in M_2 : \exists (a, t_1) \in M_1 : (t_1, t_2) \in R$ .

Again, we write  $\mathcal{E}_1 \leq_m \mathcal{E}_2$  whenever there exists such a modal refinement  $R$  such that for all  $x_1^0 \in X_1^0$ , there exists  $x_2^0 \in X_2^0$  for which  $(x_1^0, x_2^0) \in R$ . Sets of implementations and thorough refinement are defined as for DMTS.

**Structural equivalence.** We proceed to show that the three formalisms introduced in this section are structurally equivalent. Using the translations between AA and DMTS discovered in [6] and new translations between AA and hybrid logic, we show that these respect modal refinement.

The translations *al*, *la* between AA and our hybrid logic are straightforward: For an AA  $\mathcal{A} = (S, S^0, \text{Tran})$  and all  $s \in S$ , let

$$\Phi(s) = \bigvee_{M \in \text{Tran}(s)} \left( \bigwedge_{(a,t) \in M} \langle a \rangle t \wedge \bigwedge_{(b,u) \notin M} \neg \langle b \rangle u \right)$$

and define the  $\mathcal{L}$ -expression  $al(\mathcal{A}) = (S, S^0, \Phi)$ .

For an  $\mathcal{L}$ -expression  $\mathcal{E} = (X, X^0, \Phi)$  and all  $x \in X$ , let  $\text{Tran}(x) = \llbracket \Phi(x) \rrbracket$  and define the AA  $la(\mathcal{E}) = (X, X^0, \text{Tran})$ .

The translations *da*, *ad* between DMTS and AA were discovered in [6]. For a DMTS  $\mathcal{D} = (S, S^0, \dashrightarrow, \longrightarrow)$  and all  $s \in S$ , let

$$\text{Tran}(s) = \{M \subseteq \Sigma \times S \mid \forall (a, t) \in M : s \dashrightarrow^a t, \forall s \longrightarrow N : N \cap M \neq \emptyset\}$$

and define the AA  $da(\mathcal{D}) = (S, S^0, \text{Tran})$ .<sup>1</sup>

For an AA  $\mathcal{A} = (S, S^0, \text{Tran})$ , define the DMTS  $ad(\mathcal{A}) = (D, D^0, \dashrightarrow, \longrightarrow)$  as follows:

$$\begin{aligned} D &= \{M \in \text{Tran}(s) \mid s \in S\} \\ D^0 &= \{M^0 \in \text{Tran}(s^0) \mid s^0 \in S^0\} \\ \longrightarrow &= \{(M, \{(a, M') \mid M' \in \text{Tran}(t)\}) \mid (a, t) \in M\} \\ \dashrightarrow &= \{(M, a, M') \mid \exists M \longrightarrow N : (a, M') \in N\} \end{aligned}$$

Note that the state spaces of  $\mathcal{A}$  and  $ad(\mathcal{A})$  are not the same; the one of  $ad(\mathcal{A})$  may be exponentially larger. The following lemma shows that this explosion is unavoidable:

**Lemma 4.** *There exists a one-state AA  $\mathcal{A}$  for which any DMTS  $\mathcal{D} \equiv_{\text{th}} \mathcal{A}$  has at least  $2^{n-1}$  states, where  $n$  is the size of the alphabet  $\Sigma$ .*

<sup>1</sup> Note that there is an error in the corresponding formula in [6].

We notice that LTS are preserved by all translations: for any LTS  $\mathcal{I}$ ,  $al(\mathcal{I}) = la(\mathcal{I}) = da(\mathcal{I}) = ad(\mathcal{I}) = \mathcal{I}$ . In [6] it is shown that the translations between AA and DMTS respect sets of implementations, *i.e.* that  $da(\mathcal{D}) \equiv_{th} \mathcal{D}$  and  $ad(\mathcal{A}) \equiv_{th} \mathcal{A}$  for all DMTS  $\mathcal{D}$  and all AA  $\mathcal{A}$ . The next theorem shows that these and the other presented translations respect modal refinement, hence these formalisms are not only semantically equivalent, but *structurally equivalent*.

**Theorem 5.** *For all AA  $\mathcal{A}_1, \mathcal{A}_2$ , DMTS  $\mathcal{D}_1, \mathcal{D}_2$  and  $\mathcal{L}$ -expressions  $\mathcal{E}_1, \mathcal{E}_2$ :*

1.  $\mathcal{A}_1 \leq_m \mathcal{A}_2$  *iff*  $al(\mathcal{A}_1) \leq_m al(\mathcal{A}_2)$ ,
2.  $\mathcal{E}_1 \leq_m \mathcal{E}_2$  *iff*  $la(\mathcal{E}_1) \leq_m la(\mathcal{E}_2)$ ,
3.  $\mathcal{D}_1 \leq_m \mathcal{D}_2$  *iff*  $da(\mathcal{D}_1) \leq_m da(\mathcal{D}_2)$ , and
4.  $\mathcal{A}_1 \leq_m \mathcal{A}_2$  *iff*  $ad(\mathcal{A}_1) \leq_m ad(\mathcal{A}_2)$ .

*Proof (sketch).* We give a few hints about the proofs of the equivalences; the details can be found in appendix. The first two equivalences follow easily from the definitions, once one notices that for both translations,  $\llbracket \Phi(x) \rrbracket = \text{Tran}(x)$  for all  $x \in X$ . For the third equivalence, we can show that a DMTS modal refinement  $\mathcal{D}_1 \leq_m \mathcal{D}_2$  is also an AA modal refinement  $da(\mathcal{D}_1) \leq_m da(\mathcal{D}_2)$  and vice versa.

The fourth equivalence is slightly more tricky, as the state space changes. If  $R \subseteq S_1 \times S_2$  is an AA modal refinement relation witnessing  $\mathcal{A}_1 \leq_m \mathcal{A}_2$ , then we can construct a DMTS modal refinement  $R' \subseteq D_1 \times D_2$ , which witnesses  $ad(\mathcal{A}_1) \leq_m ad(\mathcal{A}_2)$ , by

$$\begin{aligned} R' = \{ (M_1, M_2) \mid \exists (s_1, s_2) \in R : M_1 \in \text{Tran}_1(s_1), M_2 \in \text{Tran}_2(s_2), \\ \forall (a, t_1) \in M_1 : \exists (a, t_2) \in M_2 : (t_1, t_2) \in R, \\ \forall (a, t_2) \in M_2 : \exists (a, t_1) \in M_1 : (t_1, t_2) \in R \}. \end{aligned}$$

Conversely, if  $R \subseteq D_1 \times D_2$  is a DMTS modal refinement witnessing  $ad(\mathcal{A}_1) \leq_m ad(\mathcal{A}_2)$ , then  $R' \subseteq S_1 \times S_2$  given by

$$R' = \{ (s_1, s_2) \mid \forall M_1 \in \text{Tran}_1(s_1) : \exists M_2 \in \text{Tran}_2(s_2) : (M_1, M_2) \in R \}$$

is an AA modal refinement. □

The result on thorough equivalence from [6] now easily follows:

**Corollary 6.** *For all AA  $\mathcal{A}$ , DMTS  $\mathcal{D}$  and  $\mathcal{L}$ -expressions  $\mathcal{E}$ ,  $al(\mathcal{A}) \equiv_{th} \mathcal{A}$ ,  $la(\mathcal{E}) \equiv_{th} \mathcal{E}$ ,  $da(\mathcal{D}) \equiv_{th} \mathcal{D}$ , and  $ad(\mathcal{A}) \equiv_{th} \mathcal{A}$ . □*

Also soundness of modal refinement for AA and hybrid logic follows directly from Theorem 5:

**Corollary 7.** *For all AA  $\mathcal{A}_1$  and  $\mathcal{A}_2$ ,  $\mathcal{A}_1 \leq_m \mathcal{A}_2$  implies  $\mathcal{A}_1 \leq_{th} \mathcal{A}_2$ . For all  $\mathcal{L}$ -expressions  $\mathcal{E}_1$  and  $\mathcal{E}_2$ ,  $\mathcal{E}_1 \leq_m \mathcal{E}_2$  implies  $\mathcal{E}_1 \leq_{th} \mathcal{E}_2$ . □*



### 3 The Modal $\nu$ -Calculus

We wish to extend the structural equivalences of the previous section to the modal  $\nu$ -calculus. Using translations between AA, DMTS and  $\nu$ -calculus based on work in [22, 11], it has been shown in [6] that  $\nu$ -calculus and DMTS/AA are *semantically* equivalent. We will see below that there is a *mismatch* between the translations from [6] (and hence between the translations in [22, 11]) which precludes structural equivalence and then proceed to propose a new translation which fixes the mismatch.

**Syntax and semantics.** We first recall the syntax and semantics of the modal  $\nu$ -calculus, the fragment of the modal  $\mu$ -calculus [33, 21] with only maximal fixed points. Instead of an explicit maximal fixed point operator, we use the representation by equation systems in Hennessy-Milner logic developed in [24].

For a finite set  $X$  of variables, let  $\mathcal{H}(X)$  be the set of *Hennessy-Milner formulae*, generated by the abstract syntax  $\mathcal{H}(X) \ni \phi ::= \mathbf{tt} \mid \mathbf{ff} \mid x \mid \langle a \rangle \phi \mid [a] \phi \mid \phi \wedge \phi \mid \phi \vee \phi$ , for  $a \in \Sigma$  and  $x \in X$ .

A *declaration* is a mapping  $\Delta : X \rightarrow \mathcal{H}(X)$ ; we recall the maximal fixed point semantics of declarations from [24]. Let  $(S, S^0, \longrightarrow)$  be an LTS, then an *assignment* is a mapping  $\sigma : X \rightarrow 2^S$ . The set of assignments forms a complete lattice with order  $\sigma_1 \sqsubseteq \sigma_2$  iff  $\sigma_1(x) \subseteq \sigma_2(x)$  for all  $x \in X$  and lowest upper bound  $(\bigsqcup_{i \in I} \sigma_i)(x) = \bigcup_{i \in I} \sigma_i(x)$ .

The semantics of a formula is a subset of  $S$ , given relative to an assignment  $\sigma$ , defined as follows:  $\langle \mathbf{tt} \rangle \sigma = S$ ,  $\langle \mathbf{ff} \rangle \sigma = \emptyset$ ,  $\langle x \rangle \sigma = \sigma(x)$ ,  $\langle \phi \wedge \psi \rangle \sigma = \langle \phi \rangle \sigma \cap \langle \psi \rangle \sigma$ ,  $\langle \phi \vee \psi \rangle \sigma = \langle \phi \rangle \sigma \cup \langle \psi \rangle \sigma$ , and

$$\begin{aligned} \langle \langle a \rangle \phi \rangle \sigma &= \{s \in S \mid \exists s \xrightarrow{a} s' : s' \in \langle \phi \rangle \sigma\}, \\ \langle [a] \phi \rangle \sigma &= \{s \in S \mid \forall s \xrightarrow{a} s' : s' \in \langle \phi \rangle \sigma\}. \end{aligned}$$

The semantics of a declaration  $\Delta$  is then the assignment defined by

$$\langle \Delta \rangle = \bigsqcup \{ \sigma : X \rightarrow 2^S \mid \forall x \in X : \sigma(x) \subseteq \langle \Delta(x) \rangle \sigma \};$$

the maximal (pre)fixed point of  $\Delta$ .

A  $\nu$ -calculus expression is a structure  $\mathcal{N} = (X, X^0, \Delta)$ , with  $X^0 \subseteq X$  sets of variables and  $\Delta : X \rightarrow \mathcal{H}(X)$  a declaration. We say that an LTS  $\mathcal{I} = (S, S^0, \longrightarrow)$  *implements* (or *models*) the expression, and write  $\mathcal{I} \models \mathcal{N}$ , if it holds that for all  $s^0 \in S^0$ , there is  $x^0 \in X^0$  such that  $s^0 \in \langle \Delta \rangle(x^0)$ . We write  $\llbracket \mathcal{N} \rrbracket$  for the set of implementations (models) of a  $\nu$ -calculus expression  $\mathcal{N}$ . As for DMTS, we write  $\llbracket x \rrbracket = \llbracket (X, \{x\}, \Delta) \rrbracket$  for  $x \in X$ , and thorough refinement of expressions and states is defined accordingly.

The following lemma introduces a *normal form* for  $\nu$ -calculus expressions:

**Lemma 8.** *For any  $\nu$ -calculus expression  $\mathcal{N}_1 = (X_1, X_1^0, \Delta_1)$ , there exists another expression  $\mathcal{N}_2 = (X_2, X_2^0, \Delta_2)$  with  $\llbracket \mathcal{N}_1 \rrbracket = \llbracket \mathcal{N}_2 \rrbracket$  and such that for any  $x \in X$ ,  $\Delta_2(x)$  is of the form*

$$\Delta_2(x) = \bigwedge_{i \in I} \left( \bigvee_{j \in J_i} \langle a_{ij} \rangle x_{ij} \right) \wedge \bigwedge_{a \in \Sigma} [a] \left( \bigvee_{j \in J_a} y_{a,j} \right) \quad (1)$$

for finite (possibly empty) index sets  $I$ ,  $J_i$ ,  $J_a$ , for  $i \in I$  and  $a \in \Sigma$ , and all  $x_{ij}, y_{a,j} \in X_2$ . Additionally, for all  $i \in I$  and  $j \in J_i$ , there exists  $j' \in J_{a_{ij}}$  for which  $x_{ij} \leq_{\text{th}} y_{a_{ij},j'}$ .

As this is a type of *conjunctive normal form*, it is clear that translating a  $\nu$ -calculus expression into normal form may incur an exponential blow-up.

We introduce some notation for  $\nu$ -calculus expressions in normal form which will make our life easier later. Let  $\mathcal{N} = (X, X^0, \Delta)$  be such an expression and  $x \in X$ , with  $\Delta(x) = \bigwedge_{i \in I} (\bigvee_{j \in J_i} \langle a_{ij} \rangle x_{ij}) \wedge \bigwedge_{a \in \Sigma} [a] (\bigvee_{j \in J_a} y_{a,j})$  as in the lemma. Define  $\diamond(x) = \{ \{ (a_{ij}, x_{ij}) \mid j \in J_i \} \mid i \in I \}$  and, for each  $a \in \Sigma$ ,  $\square^a(x) = \{ y_{a,j} \mid j \in J_a \}$ . Note that now  $\Delta(x) = \bigwedge_{N \in \diamond(x)} (\bigvee_{(a,y) \in N} \langle a \rangle y) \wedge \bigwedge_{a \in \Sigma} [a] (\bigvee_{y \in \square^a(x)} y)$ .

**Refinement.** In order to expose our structural equivalence, we need to introduce a notion of modal refinement for the modal  $\nu$ -calculus. For reasons which will become apparent later, we define two different such notions:

Let  $\mathcal{N}_1 = (X_1, X_1^0, \Delta_1)$ ,  $\mathcal{N}_2 = (X_2, X_2^0, \Delta_2)$  be  $\nu$ -calculus expressions in normal form and  $R \subseteq X_1 \times X_2$ . The relation  $R$  is a *modal refinement* if it holds for all  $(x_1, x_2) \in R$  that

1. for all  $a \in \Sigma$  and every  $y_1 \in \square_1^a(x_1)$ , there is  $y_2 \in \square_2^a(x_2)$  for which  $(y_1, y_2) \in R$ , and
2. for all  $N_2 \in \diamond_2(x_2)$  there is  $N_1 \in \diamond_1(x_1)$  such that for each  $(a, y_1) \in N_1$ , there exists  $(a, y_2) \in N_2$  with  $(y_1, y_2) \in R$ .

$R$  is a *modal-thorough refinement* if, instead of 1., it holds that

- 1'. for all  $a \in \Sigma$ , all  $y_1 \in \square_1^a(x_1)$  and every  $y'_1 \in X_1$  with  $y'_1 \leq_{\text{th}} y_1$ , there is  $y_2 \in \square_2^a(x_2)$  and  $y'_2 \in X_2$  such that  $y'_2 \leq_{\text{th}} y_2$  and  $(y'_1, y'_2) \in R$ .

We say that  $\mathcal{N}_1$  *refines*  $\mathcal{N}_2$  whenever there exists such a refinement  $R$  such that for every  $x_1^0 \in X_1^0$  there exists  $x_2^0 \in X_2^0$  for which  $(x_1^0, x_2^0) \in R$ . We write  $\mathcal{N}_1 \leq_{\text{m}} \mathcal{N}_2$  in case of modal and  $\mathcal{N}_1 \leq_{\text{mt}} \mathcal{N}_2$  in case of modal-thorough refinement.

We remark that whereas modal refinement for  $\nu$ -calculus expressions is a simple and entirely syntactic notion, modal-thorough refinement involves semantic inclusions of states. Using results in [9], this implies that modal refinement can be decided in time polynomial in the size of the (normal-form) expressions, whereas deciding modal-thorough refinement is EXPTIME-complete.

**Translation from DMTS to  $\nu$ -calculus.** Our translation from DMTS to  $\nu$ -calculus is new, but similar to the translation from AA to  $\nu$ -calculus given in [6]. This in turn is based on the *characteristic formulae* of [22] (see also [1]).

For a DMTS  $\mathcal{D} = (S, S^0, \dashv\vdash, \longrightarrow)$  and all  $s \in S$ , we define  $\diamond(s) = \{ N \mid s \longrightarrow N \}$  and, for each  $a \in \Sigma$ ,  $\square^a(s) = \{ t \mid s \dashv\vdash_a t \}$ . Then, let

$$\Delta(s) = \bigwedge_{N \in \diamond(s)} \left( \bigvee_{(a,t) \in N} \langle a \rangle t \right) \wedge \bigwedge_{a \in \Sigma} [a] \left( \bigvee_{t \in \square^a(s)} t \right)$$

and define the (normal-form)  $\nu$ -calculus expression  $dh(\mathcal{D}) = (S, S^0, \Delta)$ .

Note how the formula precisely expresses that we demand at least one of every choice of disjunctive must-transitions (first part) and permit all may-transitions (second part); this is also the intuition of the characteristic formulae of [22]. Using results of [6] (which introduces a very similar translation from AA to  $\nu$ -calculus expressions), we see that  $dh(\mathcal{D}) \equiv_{\text{th}} \mathcal{D}$  for all DMTS  $\mathcal{D}$ .

**Theorem 9.** *For all DMTS  $\mathcal{D}_1$  and  $\mathcal{D}_2$ ,  $\mathcal{D}_1 \leq_m \mathcal{D}_2$  iff  $dh(\mathcal{D}_1) \leq_m dh(\mathcal{D}_2)$ .*

*Proof.* For the forward direction, let  $R \subseteq S_1 \times S_2$  be a modal refinement between  $\mathcal{D}_1 = (S_1, S_1^0, \dashv\vdash_1, \longrightarrow_1)$  and  $\mathcal{D}_2 = (S_2, S_2^0, \dashv\vdash_2, \longrightarrow_2)$ ; we show that  $R$  is also a modal refinement between  $dh(\mathcal{D}_1) = (S_1, S_1^0, \Delta_1)$  and  $dh(\mathcal{D}_2) = (S_2, S_2^0, \Delta_2)$ . Let  $(s_1, s_2) \in R$ .

- Let  $a \in \Sigma$  and  $t_1 \in \Box_1^a(s_1)$ , then  $s_1 \dashv\vdash_1^a t_1$ , which implies that there is  $t_2 \in S_2$  for which  $s_2 \dashv\vdash_2^a t_2$  and  $(t_1, t_2) \in R$ . By definition of  $\Box_2^a$ ,  $t_2 \in \Box_2^a(s_2)$ .
- Let  $N_2 \in \Diamond_2(s_2)$ , then  $s_2 \longrightarrow_2 N_2$ , which implies that there exists  $s_1 \longrightarrow_1 N_1$  such that  $\forall(a, t_1) \in N_1 : \exists(a, t_2) \in N_2 : (t_1, t_2) \in R$ . By definition of  $\Box_1^a$ ,  $N_1 \in \Box_1^a(s_1)$ .

For the other direction, let  $R \subseteq S_1 \times S_2$  be a modal refinement between  $dh(\mathcal{D}_1)$  and  $dh(\mathcal{D}_2)$ , we show that  $R$  is also a modal refinement between  $\mathcal{D}_1$  and  $\mathcal{D}_2$ . Let  $(s_1, s_2) \in R$ .

- For all  $s_1 \dashv\vdash_1^a t_1$ ,  $t_1 \in \Box_1^a(s_1)$ , which implies that there is  $t_2 \in \Box_2^a(s_2)$  with  $(t_1, t_2) \in R$ , and by definition of  $\Box_2^a$ ,  $s_2 \dashv\vdash_2^a t_2$ .
- For all  $s_2 \longrightarrow_2 N_2$ ,  $N_2 \in \Diamond_2(s_2)$ , which implies that there is  $N_1 \in \Diamond_1(s_1)$  such that  $\forall(a, t_1) \in N_1 : \exists(a, t_2) \in N_2 : (t_1, t_2) \in R$ , and by definition of  $\Box_1^a$ ,  $s_1 \longrightarrow_1 N_1$ .  $\square$

**Old translation from  $\nu$ -calculus to DMTS.** We recall the translation from  $\nu$ -calculus to DMTS given in [6], which is based on a translation from Hennessy-Milner formulae (without recursion and fixed points) to sets of acyclic MTS in [11]. For a  $\nu$ -calculus expression  $\mathcal{N} = (X, X^0, \Delta)$  in normal form, let

$$\begin{aligned} \dashv\vdash &= \{(x, a, y') \in X \times \Sigma \times X \mid \exists y \in \Box^a(x) : y' \leq_{\text{th}} y\}, \\ \longrightarrow &= \{(x, N) \mid x \in X, N \in \Diamond(x)\}. \end{aligned}$$

and define the DMTS  $hd_t(\mathcal{N}) = (X, X^0, \dashv\vdash, \longrightarrow)$ .

Note how this translates diamonds to disjunctive must-transitions directly, but for boxes takes semantic inclusions into account: for a subformula  $[a]y$ , may-transitions are created to all variables which are semantically below  $y$ . This is consistent with the interpretation of formulae-as-properties:  $[a]y$  means “for any  $a$ -transition,  $\Delta(y)$  must hold”; but  $\Delta(y)$  holds for all variables which are semantically below  $y$ .

It follows from results in [6] (which uses a slightly different normal form for  $\nu$ -calculus expressions) that  $hd_t(\mathcal{N}) \equiv_{\text{th}} \mathcal{N}$  for all  $\nu$ -calculus expressions  $\mathcal{N}$ .

**Theorem 10.** *For all  $\nu$ -calculus expressions,  $\mathcal{N}_1 \leq_{\text{mt}} \mathcal{N}_2$  iff  $hd_t(\mathcal{N}_1) \leq_m hd_t(\mathcal{N}_2)$ .*

*Proof.* For the forward direction, let  $R \subseteq X_1 \times X_2$  be a modal-thorough refinement between  $\mathcal{N}_1 = (X_1, X_1^0, \Delta_1)$  and  $\mathcal{N}_2 = (X_2, X_2^0, \Delta_2)$ . We show that  $R$  is also a modal refinement between  $hd_t(\mathcal{N}_1) = (X_1, X_1^0, \dashrightarrow_1, \longrightarrow_2)$  and  $hd_t(\mathcal{N}_2) = (X_2, X_2^0, \dashrightarrow_2, \longrightarrow_2)$ . Let  $(x_1, x_2) \in R$ .

- Let  $x_1 \dashrightarrow_1^a y'_1$ . By definition of  $\dashrightarrow_1$ , there is  $y_1 \in \Box_1^a(x_1)$  for which  $y'_1 \leq_{th} y_1$ . Then by modal-thorough refinement, this implies that there exists  $y_2 \in \Box_2^a(x_2)$  and  $y'_2 \in X_2$  such that  $y'_2 \leq_{th} y_2$  and  $(y'_1, y'_2) \in R$ . By definition of  $\dashrightarrow_2$  we have  $x_2 \dashrightarrow_2^a y'_2$ .
- Let  $x_2 \longrightarrow_2 N_2$ , then we have  $N_2 \in \Diamond_2(x_2)$ . By modal-thorough refinement, this implies that there is  $N_1 \in \Diamond_1(x_1)$  such that  $\forall (a, y_1) \in N_1 : \exists (a, y_2) \in N_2 : (y_1, y_2) \in R$ . By definition of  $\longrightarrow_1$ ,  $x_1 \longrightarrow_1 N_1$ .

Now to the proof that  $hd_t(\mathcal{N}_1) \leq_m hd_t(\mathcal{N}_2)$  implies  $\mathcal{N}_1 \leq_{mt} \mathcal{N}_2$ . We have a modal refinement (in the DMTS sense)  $R \subseteq X_1 \times X_2$ . We must show that  $R$  is also a modal-thorough refinement. Let  $(x_1, x_2) \in R$ .

- Let  $a \in \Sigma$ ,  $y_1 \in \Box_1^a(x_1)$  and  $y'_1 \in X_1$  such that  $y'_1 \leq_{th} y_1$ . Then by definition of  $\dashrightarrow_1$ ,  $x_1 \dashrightarrow_1^a y'_1$ . By modal refinement, this implies that there exists  $x_2 \dashrightarrow_2^a y'_2$  with  $(y'_1, y'_2) \in R$ . Finally, by definition of  $\dashrightarrow_2$ , there exists  $y_2 \in \Box_2^a(x_2)$  such that  $y'_2 \leq_{th} y_2$ .
- Let  $N_2 \in \Diamond_2(x_2)$ , then by definition of  $\longrightarrow_2$ ,  $x_2 \longrightarrow_2 N_2$ . Then, by modal refinement, this implies that there exists  $x_1 \longrightarrow_1 N_1$  such that  $\forall (a, y_1) \in N_1 : \exists (a, y_2) \in N_2 : (y_1, y_2) \in R$ . By definition of  $\longrightarrow_1$ ,  $N_1 \in \Box_1^a(x_1)$ .  $\square$

**Discussion.** Notice how Theorems 9 and 10 expose a *mismatch* between the translations:  $dh$  relates DMTS refinement to  $\nu$ -calculus *modal* refinement, whereas  $hd_t$  relates it to *modal-thorough* refinement. Both translations are well-grounded in the literature and well-understood, cf. [6, 11, 22], but this mismatch has not been discovered up to now. Given that the above theorems can be understood as universal properties of the translations, it means that there is no notion of refinement for  $\nu$ -calculus which is consistent with them both.

The following lemma, easily shown by inspection, shows that this discrepancy is related to the may-completion for DMTS:

**Lemma 11.** *For any DMTS  $\mathcal{D}$ ,  $mc(\mathcal{D}) = hd_t(dh(\mathcal{D}))$ .*  $\square$

As a corollary, we see that modal refinement and modal-thorough refinement for  $\nu$ -calculus are incomparable: Referring back to Example 3, we have  $\mathcal{D} \leq_m \mathcal{D}_1$ , hence by Theorem 9,  $dh(\mathcal{D}) \leq_m dh(\mathcal{D}_1)$ . On the other hand, we know that  $mc(\mathcal{D}) \not\leq_m mc(\mathcal{D}_1)$ , i.e. by Lemma 11,  $hd_t(dh(\mathcal{D})) \not\leq_m hd_t(dh(\mathcal{D}_1))$ , and then by Theorem 10,  $dh(\mathcal{D}) \not\leq_{mt} dh(\mathcal{D}_1)$ .

To expose an example where modal-thorough refinement holds, but modal refinement does not, we note that  $mc(\mathcal{D}_2) \leq_m mc(\mathcal{D})$  implies, again using Lemma 11 and Theorem 10, that  $dh(\mathcal{D}_2) \leq_{mt} dh(\mathcal{D})$ . On the other hand, we know that  $\mathcal{D}_2 \not\leq_m \mathcal{D}$ , so by Theorem 9,  $dh(\mathcal{D}_2) \not\leq_m dh(\mathcal{D})$ .

**New translation from  $\nu$ -calculus to DMTS.** We now show that the mismatch between DMTS and  $\nu$ -calculus expressions can be fixed by introducing a new, simpler translation from  $\nu$ -calculus to DMTS.

For a  $\nu$ -calculus expression  $\mathcal{N} = (X, X^0, \Delta)$  in normal form, let

$$\begin{aligned} \dashrightarrow &= \{(x, a, y) \in X \times \Sigma \times X \mid y \in \Box^a(x)\}, \\ \longrightarrow &= \{(x, N) \mid x \in X, N \in \Diamond(x)\}. \end{aligned}$$

and define the DMTS  $hd(\mathcal{N}) = (X, X^0, \dashrightarrow, \longrightarrow)$ . This is a simple syntactic translation: boxes are translated to disjunctive must-transitions and diamonds to may-transitions.

**Theorem 12.** *For all  $\nu$ -calculus expressions,  $\mathcal{N}_1 \leq_m \mathcal{N}_2$  iff  $hd(\mathcal{N}_1) \leq_m hd(\mathcal{N}_2)$ .*

*Proof.* Let  $R \subseteq X_1 \times X_2$  be a modal refinement between  $\mathcal{N}_1 = (X_1, X_1^0, \Delta_1)$  and  $\mathcal{N}_2 = (X_2, X_2^0, \Delta_2)$ ; we show that  $R$  is also a modal refinement between  $hd(\mathcal{N}_1) = (S_1, S_1^0, \dashrightarrow_1, \longrightarrow_1)$  and  $hd(\mathcal{N}_2) = (S_2, S_2^0, \dashrightarrow_2, \longrightarrow_2)$ . Let  $(x_1, x_2) \in R$ .

- Let  $x_1 \dashrightarrow_1^a y_1$ , then  $y_1 \in \Box_1^a(x_1)$ , which implies that there exists  $y_2 \in \Box_2^a(x_2)$  for which  $(y_1, y_2) \in R$ , and by definition of  $\dashrightarrow_2$ ,  $x_2 \dashrightarrow_2^a y_2$ .
- Let  $x_2 \longrightarrow_2 N_2$ , then  $N_2 \in \Diamond_2(x_2)$ , hence there is  $N_1 \in \Diamond_1(x_1)$  such that  $\forall(a, y_1) \in N_1 : \exists(a, y_2) \in N_2 : (y_1, y_2) \in R$ , and by definition of  $\longrightarrow_1$ ,  $x_1 \longrightarrow_1 N_1$ .

Now let  $R \subseteq X_1 \times X_2$  be a modal refinement between  $hd(\mathcal{N}_1)$  and  $hd(\mathcal{N}_2)$ , we show that  $R$  is also a modal refinement between  $\mathcal{N}_1$  and  $\mathcal{N}_2$ . Let  $(x_1, x_2) \in R$ ,

- Let  $a \in \Sigma$  and  $y_1 \in \Box_1^a(x_1)$ . Then  $x_1 \dashrightarrow_1^a y_1$ , which implies that there is  $y_2 \in X_2$  for which  $x_2 \dashrightarrow_2^a y_2$  and  $(y_1, y_2) \in R$ , and by definition of  $\dashrightarrow_2$ ,  $t_2 \in \Box_2^a(s_2)$ .
- Let  $N_2 \in \Diamond_2(x_2)$ , then  $x_2 \longrightarrow_2 N_2$ , so there is  $x_1 \longrightarrow_1 N_1$  such that  $\forall(a, y_1) \in N_1 : \exists(a, y_2) \in N_2 : (y_1, y_2) \in R$ . By definition of  $\longrightarrow_1$ ,  $N_1 \in \Box_1^a(x_1)$ .  $\square$

We finish the section by proving that also for the syntactic translation  $hd(\mathcal{N}) \equiv_{th} \mathcal{N}$  for all  $\nu$ -calculus expressions; this shows that our translation can serve as a replacement for the partly-semantic  $hd_t$  translation from [6, 11]. First we remark that  $dh$  and  $hd$  are inverses to each other:

**Proposition 13.** *For any  $\nu$ -calculus expression  $\mathcal{N}$ ,  $dh(hd(\mathcal{N})) = \mathcal{N}$ ; for any DMTS  $\mathcal{D}$ ,  $hd(dh(\mathcal{D})) = \mathcal{D}$ .*  $\square$

**Corollary 14.** *For all  $\nu$ -calculus expressions  $\mathcal{N}$ ,  $hd(\mathcal{N}) \equiv_{th} \mathcal{N}$ .*  $\square$

## 4 The Modal $\nu$ -Calculus as a Specification Theory

Now that we have exposed a close structural correspondence between the modal  $\nu$ -calculus and DMTS, we can transfer the operations which make DMTS a complete specification theory to the  $\nu$ -calculus.

**Refinement and implementations.** As for DMTS and AA, we can define an embedding of LTS into the modal  $\nu$ -calculus so that implementation  $\models$  and refinement  $\leq_m$  coincide. We say that a  $\nu$ -calculus expression  $(X, X^0, \Delta)$  in normal form is an *implementation* if  $\Diamond(x) = \{\{(a, y)\} \mid y \in \Box^a(x), a \in \Sigma\}$  for all  $x \in X$ .

The  $\nu$ -calculus translation of a LTS  $(S, S^0, \longrightarrow)$  is the expression  $(S, S^0, \Delta)$  in normal form with  $\Diamond(s) = \{\{(a, t)\} \mid s \xrightarrow{a} t\}$  and  $\Box^a(s) = \{t \mid s \xrightarrow{a} t\}$ . This defines a bijection between LTS and  $\nu$ -calculus implementations.

**Theorem 15.** *For any LTS  $\mathcal{I}$  and any  $\nu$ -calculus expression  $\mathcal{N}$ ,  $\mathcal{I} \models \mathcal{N}$  iff  $\mathcal{I} \leq_m \mathcal{N}$ .*

*Proof.*  $\mathcal{I} \models \mathcal{N}$  is the same as  $\mathcal{I} \in \llbracket \mathcal{N} \rrbracket$ , which by Corollary 14 is equivalent to  $\mathcal{I} \in \llbracket hd(\mathcal{N}) \rrbracket$ . By definition, this is the same as  $\mathcal{I} \leq_m hd(\mathcal{N})$ , which using Theorem 12 is equivalent to  $\mathcal{I} \leq_m \mathcal{N}$ .  $\square$

Using transitivity, this implies that modal refinement for  $\nu$ -calculus is sound:

**Corollary 16.** *For all  $\nu$ -calculus expressions,  $\mathcal{N}_1 \leq_m \mathcal{N}_2$  implies  $\mathcal{N}_1 \leq_{th} \mathcal{N}_2$ .  $\square$*

**Disjunction and conjunction.** As for DMTS, disjunction of  $\nu$ -calculus expressions is straight-forward. Given  $\nu$ -calculus expressions  $\mathcal{N}_1 = (X_1, X_1^0, \Delta_1)$ ,  $\mathcal{N}_2 = (X_2, X_2^0, \Delta_2)$  in normal form, their *disjunction* is  $\mathcal{N}_1 \vee \mathcal{N}_2 = (X_1 \cup X_2, X_1^0 \cup X_2^0, \Delta)$  with  $\Delta(x_1) = \Delta_1(x_1)$  for  $x_1 \in X_1$  and  $\Delta(x_2) = \Delta_2(x_2)$  for  $x_2 \in X_2$ .

The *conjunction* of  $\nu$ -calculus expressions like above is  $\mathcal{N}_1 \wedge \mathcal{N}_2 = (X, X^0, \Delta)$  defined by  $X = X_1 \times X_2$ ,  $X^0 = X_1^0 \times X_2^0$ ,  $\Box^a(x_1, x_2) = \Box_1^a(x_1) \times \Box_2^a(x_2)$  for each  $(x_1, x_2) \in X$ ,  $a \in \Sigma$ , and for each  $(x_1, x_2) \in X$ ,

$$\begin{aligned} \Diamond(x_1, x_2) = & \{ \{(a, (y_1, y_2))\} \mid (a, y_1) \in N_1, (y_1, y_2) \in \Box^a(x_1, x_2) \} \mid N_1 \in \Diamond_1(x_1) \} \\ & \cup \{ \{(a, (y_1, y_2))\} \mid (a, y_2) \in N_2, (y_1, y_2) \in \Box^a(x_1, x_2) \} \mid N_2 \in \Diamond_2(x_2) \}. \end{aligned}$$

Note that both  $\mathcal{N}_1 \vee \mathcal{N}_2$  and  $\mathcal{N}_1 \wedge \mathcal{N}_2$  are again  $\nu$ -calculus expressions in normal form.

**Theorem 17.** *For all  $\nu$ -calculus expressions  $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$  in normal form,*

- $\mathcal{N}_1 \vee \mathcal{N}_2 \leq_m \mathcal{N}_3$  iff  $\mathcal{N}_1 \leq_m \mathcal{N}_3$  and  $\mathcal{N}_2 \leq_m \mathcal{N}_3$ ,
- $\mathcal{N}_1 \leq_m \mathcal{N}_2 \wedge \mathcal{N}_3$  iff  $\mathcal{N}_1 \leq_m \mathcal{N}_2$  and  $\mathcal{N}_1 \leq_m \mathcal{N}_3$ ,
- $\llbracket \mathcal{N}_1 \vee \mathcal{N}_2 \rrbracket = \llbracket \mathcal{N}_1 \rrbracket \cup \llbracket \mathcal{N}_2 \rrbracket$ , and  $\llbracket \mathcal{N}_1 \wedge \mathcal{N}_2 \rrbracket = \llbracket \mathcal{N}_1 \rrbracket \cap \llbracket \mathcal{N}_2 \rrbracket$ .

**Theorem 18.** *With operations  $\vee$  and  $\wedge$ , the class of  $\nu$ -calculus expressions forms a bounded distributive lattice up to  $\equiv_m$ .*

The bottom element (up to  $\equiv_m$ ) in the lattice is the empty  $\nu$ -calculus expression  $\perp = (\emptyset, \emptyset, \emptyset)$ , and the top element (up to  $\equiv_m$ ) is  $\top = (\{s\}, \{s\}, \Delta)$  with  $\Delta(s) = \mathbf{tt}$ .

**Structural composition.** The structural composition operator for a specification theory is to mimic, at specification level, the structural composition of implementations. That is to say, if  $\parallel$  is a composition operator for implementations (LTS), then the goal is to extend  $\parallel$  to specifications such that for all specifications  $\mathcal{S}_1, \mathcal{S}_2$ ,

$$\llbracket \mathcal{S}_1 \parallel \mathcal{S}_2 \rrbracket = \{ \mathcal{I}_1 \parallel \mathcal{I}_2 \mid \mathcal{I}_1 \in \llbracket \mathcal{S}_1 \rrbracket, \mathcal{I}_2 \in \llbracket \mathcal{S}_2 \rrbracket \}. \quad (2)$$

For simplicity, we use CSP-style synchronization for structural composition of LTS, however, our results readily carry over to other types of composition. Analogously to the situation for MTS [8], we have the following negative result:

**Theorem 19.** *There is no operator  $\parallel$  for the  $\nu$ -calculus which satisfies (2).*

*Proof.* We first note that due to Theorem 17, it is the case that implementation sets of  $\nu$ -calculus expressions are closed under disjunction: for any  $\nu$ -calculus expression  $\mathcal{N}$  and  $\mathcal{I}_1, \mathcal{I}_2 \in \llbracket \mathcal{N} \rrbracket$ , also  $\mathcal{I}_1 \vee \mathcal{I}_2 \in \llbracket \mathcal{N} \rrbracket$ .

Now assume there were an operator as in the theorem, then because of the translations, (2) would also hold for DMTS. Hence for all DMTS  $\mathcal{D}_1, \mathcal{D}_2$ ,  $\{ \mathcal{I}_1 \parallel \mathcal{I}_2 \mid \mathcal{I}_1 \in \llbracket \mathcal{D}_1 \rrbracket, \mathcal{I}_2 \in \llbracket \mathcal{D}_2 \rrbracket \}$  would be closed under disjunction. But Example 7.8 in [8] exhibits two DMTS (actually, MTS) for which this is not the case, a contradiction.  $\square$

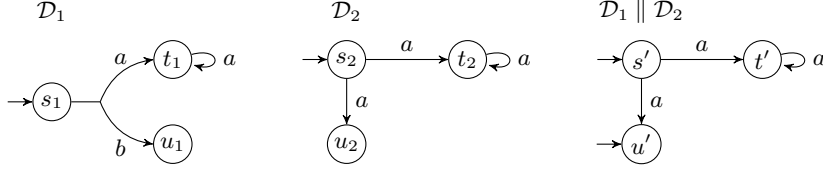
Given that we cannot have (2), the revised goal is to have a *sound* composition operator for which the right-to-left inclusion holds in (2). We can obtain one such from the structural composition of AA introduced in [6]. We hence define, for  $\nu$ -calculus expressions  $\mathcal{N}_1 = (X_1, X_1^0, \Delta_1)$ ,  $\mathcal{N}_2 = (X_2, X_2^0, \Delta_2)$  in normal form,  $\mathcal{N}_1 \parallel \mathcal{N}_2 = ah(ha(\mathcal{N}_1) \parallel_A ha(\mathcal{N}_2))$ , where  $\parallel_A$  is AA composition and we write  $ah = dh \circ ad$  and  $ha = da \circ hd$  for the composed translations.

Notice that the involved translation from AA to DMTS may lead to an exponential blow-up. Unraveling the definition gives us the following explicit expression for  $\mathcal{N}_1 \parallel \mathcal{N}_2 = (X, X^0, \Delta)$ :

- $X = \{ \{ (a, (y_1, y_2)) \mid \forall i \in \{1, 2\} : (a, y_i) \in M_i \} \mid \forall i \in \{1, 2\} : M_i \subseteq \Sigma \times X_i, \exists x_i \in X_i : \forall (a, y'_i) \in M_i : y'_i \in \Box_i^a(x_i), \forall N_i \in \Diamond_i(x_i) : N_i \cap M_i \neq \emptyset \} ,$
- $X^0 = \{ \{ (a, (y_1, y_2)) \mid \forall i \in \{1, 2\} : (a, y_i) \in M_i \} \mid \forall i \in \{1, 2\} : M_i \subseteq \Sigma \times X_i, \exists x_i \in X_i^0 : \forall (a, y'_i) \in M_i : y'_i \in \Box_i^a(x_i), \forall N_i \in \Diamond_i(x_i) : N_i \cap M_i \neq \emptyset \} ,$
- $\Diamond(x) = \{ \{ (a, \{ (b, (z_1, z_2)) \mid \forall i \in \{1, 2\} : (b, z_i) \in M_i \} \mid \forall i \in \{1, 2\} : M_i \subseteq \Sigma \times X_i, \forall (a, z'_i) \in M_i : z'_i \in \Box_i^b(y_i), \forall N_i \in \Diamond_i(y_i) : N_i \cap M_i \neq \emptyset \} \mid (a, (y_1, y_2)) \in x \} \text{ for each } x \in X, \text{ and}$
- $\Box^a(x) = \{ y \mid \exists N \in \Diamond(x) : (a, y) \in N \}.$

**Theorem 20.** *For all  $\nu$ -calculus expressions  $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$  in normal form,  $\mathcal{N}_1 \leq_m \mathcal{N}_3$  and  $\mathcal{N}_2 \leq_m \mathcal{N}_4$  imply  $\mathcal{N}_1 \parallel \mathcal{N}_2 \leq_m \mathcal{N}_3 \parallel \mathcal{N}_4$ .*

*Proof.* This follows directly from the analogous property for AA [6] and the translation theorems 5, 9 and 12.  $\square$



**Fig. 4.** DMTS  $\mathcal{D}_1$ ,  $\mathcal{D}_2$  and the reachable parts of their structural composition  $\mathcal{D}_1 \parallel \mathcal{D}_2$ . Here,  $s' = \{(a, (t_1, t_2)), (a, (t_1, u_2))\}$ ,  $t' = \{(a, (t_1, t_2))\}$  and  $u' = \emptyset$ . Note that  $\mathcal{D}_1 \parallel \mathcal{D}_2$  has two initial states.

This implies the right-to-left inclusion in (2), *i.e.*  $\{\mathcal{I}_1 \parallel \mathcal{I}_2 \mid \mathcal{I}_1 \in \llbracket \mathcal{N}_1 \rrbracket, \mathcal{I}_2 \in \llbracket \mathcal{N}_2 \rrbracket\} \subseteq \llbracket \mathcal{N}_1 \parallel \mathcal{N}_2 \rrbracket$ . It also entails *independent implementability*, in that the structural composition of the two refined specifications  $\mathcal{N}_1, \mathcal{N}_2$  is a refinement of the composition of the original specifications  $\mathcal{N}_3, \mathcal{N}_4$ . Fig. 4 shows an example of the DMTS analogue of this structural composition.

**Quotient.** The quotient operator  $/$  for a specification theory is used to synthesize specifications for components of a structural composition. Hence it is to have the property, for all specifications  $\mathcal{S}, \mathcal{S}_1$  and all implementations  $\mathcal{I}_1, \mathcal{I}_2$ , that

$$\mathcal{I}_1 \in \llbracket \mathcal{S}_1 \rrbracket \text{ and } \mathcal{I}_2 \in \llbracket \mathcal{S} / \mathcal{S}_1 \rrbracket \text{ imply } \mathcal{I}_1 \parallel \mathcal{I}_2 \in \llbracket \mathcal{S} \rrbracket. \quad (3)$$

Furthermore,  $\mathcal{S} / \mathcal{S}_1$  is to be as permissive as possible.

We can again obtain such a quotient operator for  $\nu$ -calculus from the one for AA introduced in [6]. Hence we define, for  $\nu$ -calculus expressions  $\mathcal{N}_1, \mathcal{N}_2$  in normal form,  $\mathcal{N}_1 / \mathcal{N}_2 = ah(ha(\mathcal{N}_1) /_{\mathbf{A}} ha(\mathcal{N}_2))$ , where  $/_{\mathbf{A}}$  is AA quotient. We recall the construction of  $/_{\mathbf{A}}$  from [6]:

Let  $\mathcal{A}_1 = (S_1, S_1^0, \text{Tran}_1)$ ,  $\mathcal{A}_2 = (S_2, S_2^0, \text{Tran}_2)$  be AA and define  $\mathcal{A}_1 /_{\mathbf{A}} \mathcal{A}_2 = (S, \{s^0\}, \text{Tran})$ , with  $S = 2^{S_1 \times S_2}$ ,  $s^0 = \{(s_1^0, s_2^0) \mid s_1^0 \in S_1^0, s_2^0 \in S_2^0\}$ , and  $\text{Tran}$  given as follows:

Let  $\text{Tran}(\emptyset) = 2^{\Sigma \times \{\emptyset\}}$ . For  $s = \{(s_1^1, s_2^1), \dots, (s_1^n, s_2^n)\} \in S$ , say that  $a \in \Sigma$  is *permissible from*  $s$  if it holds for all  $i = 1, \dots, n$  that there is  $M_1 \in \text{Tran}_1(s_1^i)$  and  $t_1 \in S_1$  for which  $(a, t_1) \in M_1$ , or else there is no  $M_2 \in \text{Tran}_2(s_2^i)$  and no  $t_2 \in S_2$  for which  $(a, t_2) \in M_2$ .

For  $a$  permissible from  $s$  and  $i \in \{1, \dots, n\}$ , let  $\{t_2^{i,1}, \dots, t_2^{i,m_i}\} = \{t_2 \in S_2 \mid \exists M_2 \in \text{Tran}_2(s_2^i) : (a, t_2) \in M_2\}$  be an enumeration of the possible states in  $S_2$  after an  $a$ -transition and define  $pt_a(s) = \{\{(t_1^{i,j}, t_2^{i,j}) \mid i = 1, \dots, n, j = 1, \dots, m_i\} \mid \forall i : \forall j : \exists M_1 \in \text{Tran}_1(s_1^i) : (a, t_1^{i,j}) \in M_1\}$ , the set of all sets of possible assignments of next- $a$  states from  $s_1^i$  to next- $a$  states from  $s_2^i$ .

Now let  $pt(s) = \{(a, t) \mid t \in pt_a(s), a \text{ admissible from } s\}$  and define  $\text{Tran}(s) = \{M \subseteq pt(s) \mid \forall i = 1, \dots, n : \forall M_2 \in \text{Tran}_2(s_2^i) : M \triangleright M_2 \in \text{Tran}_1(s_1^i)\}$ . Here  $\triangleright$  is the composition-projection operator defined by  $M \triangleright M_2 = \{(a, t \triangleright t_2) \mid (a, t) \in M, (a, t_2) \in M_2\}$  and  $t \triangleright t_2 = \{(t_1^1, t_2^1), \dots, (t_1^k, t_2^k)\} \triangleright t_2^i = t_1^i$  (note that by construction, there is precisely one pair in  $t$  whose second component is  $t_2^i$ ).



**Theorem 21.** *For all  $\nu$ -calculus expressions  $\mathcal{N}, \mathcal{N}_1, \mathcal{N}_2$  in normal form,  $\mathcal{N}_2 \leq_m \mathcal{N} / \mathcal{N}_1$  iff  $\mathcal{N}_1 \parallel \mathcal{N}_2 \leq_m \mathcal{N}$ .*

*Proof.* From the analogous property for AA [6] and Theorems 5, 9 and 12.  $\square$

As a corollary, we get (3): If  $\mathcal{I}_2 \in \llbracket \mathcal{N} / \mathcal{N}_1 \rrbracket$ , i.e.  $\mathcal{I}_2 \leq_m \mathcal{N} / \mathcal{N}_1$ , then  $\mathcal{N}_1 \parallel \mathcal{I}_2 \leq_m \mathcal{N}$ , which using  $\mathcal{I}_1 \leq_m \mathcal{N}_1$  and Theorem 20 implies  $\mathcal{I}_1 \parallel \mathcal{I}_2 \leq_m \mathcal{N}_1 \parallel \mathcal{I}_2 \leq_m \mathcal{N}$ . The reverse implication in Theorem 21 implies that  $\mathcal{N} / \mathcal{N}_1$  is as permissive as possible.

**Theorem 22.** *With operations  $\wedge, \vee, \parallel$  and  $/$ , the class of  $\nu$ -calculus expressions forms a commutative residuated lattice up to  $\equiv_m$ .*

The unit of  $\parallel$  (up to  $\equiv_m$ ) is the  $\nu$ -calculus expression corresponding to the LTS  $U = (\{u\}, \{u\}, \{(u, a, u) \mid a \in \Sigma\})$ . We refer to [19] for a good reference on commutative residuated lattices.

## 5 Conclusion and Further Work

Using new translations between the modal  $\nu$ -calculus and DMTS, we have exposed a structural equivalence between these two specification formalisms. This means that both types of specifications can be freely mixed; there is no more any need to decide, whether due to personal preference or for technical reasons, between one and the other. Of course, the modal  $\nu$ -calculus can only express safety properties; for more expressivity, one has to turn to more expressive logics, and no behavioral analogue to these stronger logics is known (neither is it likely to exist, we believe).

Our constructions of composition and quotient for the modal  $\nu$ -calculus expect (and return)  $\nu$ -calculus expressions in normal form, and it is an interesting question whether they can be defined for general  $\nu$ -calculus expressions. (For disjunction and conjunction this is of course trivial.) Larsen's [23] has composition and quotient operators for Hennessy-Milner logic (restricted to “deterministic context systems”), but we know of no extension (other than ours) to more general logics.

We also note that our hybrid modal logic appears related to the *Boolean equation systems* [27, 25] which are used in some  $\mu$ -calculus model checking algorithms. The precise relation between the modal  $\nu$ -calculus, our  $\mathcal{L}$ -expressions and Boolean equation systems should be worked out. Similarly, acceptance automata bear some similarity to the *modal automata* of [12].

Lastly, we should note that we have in [4, 3] introduced *quantitative* specification theories for weighted modal transition systems. These are well-suited for specification and analysis of systems with quantitative information, in that they replace the standard Boolean notion of refinement with a robust distance-based notion. We are working on an extension of these quantitative formalisms to DMTS, and hence to the modal  $\nu$ -calculus, which should relate our work to other approaches at quantitative model checking such as e.g. [17, 16, 18].

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## Appendix: Extra Lemmas and Proofs

**Lemma 23.** *Let  $\mathcal{D} = (S, S^0, \dashrightarrow, \longrightarrow)$  be a DMTS and  $s \in S$ . For all  $M_1, M_2 \in \text{Tran}(s)$  and all  $M \subseteq \Sigma \times S$  with  $M_1 \subseteq M \subseteq M_1 \cup M_2$ , also  $M \in \text{Tran}(s)$ .*

*Proof.* For  $i = 1, 2$ , since  $M_i \in \text{Tran}(s)$ , we know that

- for all  $(a, t) \in M_i$ ,  $(s, a, t) \in \dashrightarrow$ , and
- for all  $(s, N) \in \longrightarrow$ , there is  $(a, t) \in M_i \cap N$ .

Now as  $M \subseteq M_1 \cup M_2$ , it directly follows that for all  $(a, t) \in M$ , we have  $(s, a, t) \in \dashrightarrow$ . Moreover, since  $M_1 \subseteq M$ , we also have that for all  $(s, N) \in \longrightarrow$ , there exists  $(a, t) \in M \cap N$ . As a consequence,  $M \in \text{Tran}(s)$ .  $\square$

*Proof (of Lemma 4).* Let  $\Sigma = \{a_1, \dots, a_n\}$  and  $\mathcal{A} = (\{s^0\}, \{s^0\}, \text{Tran})$  the AA with  $\text{Tran}(s^0) = \{M \subseteq \Sigma \times \{s^0\} \mid \exists k : |M| = 2k\}$  the transition constraint containing all disjunctive choices of even cardinality. Let  $\mathcal{D} = (T, T^0, \dashrightarrow, \longrightarrow)$  be a DMTS with  $c\mathcal{D} \equiv_{\text{th}} \mathcal{A}$ ; we claim that  $\mathcal{D}$  must have at least  $2^{n-1}$  initial states.

Assume, for the purpose of contradiction, that  $T^0 = \{t_1^0, \dots, t_m^0\}$  with  $m < 2^{n-1}$ . We must have  $\bigcup_{i=1}^m \text{Tran}_T(t_i^0) = \{M \subseteq \Sigma \times T \mid \exists k : |M| = 2k\}$ , so that there is an index  $j \in \{1, \dots, m\}$  for which  $\text{Tran}_T(t_j^0) = \{M_1, M_2\}$  contains two different disjunctive choices from  $\text{Tran}_S(s^0)$ . By Lemma 23, also  $M \in \text{Tran}_T(t_j^0)$  for any  $M$  with  $M_1 \subseteq M \subseteq M_1 \cup M_2$ . But  $M_1 \cup M_2$  has greater cardinality than  $M_1$ , so that there will be an  $M \in \text{Tran}_T(t_j^0)$  with odd cardinality.  $\square$

*Proof (of Theorem 5).* The first two equivalences in the theorem follow directly from the definitions. Indeed, for the translation from  $\mathcal{L}$ -expressions to AA, we have  $\langle\!\langle \Phi(x) \rangle\!\rangle = \text{Tran}(x)$  by definition, hence  $M \in \langle\!\langle \Phi(x) \rangle\!\rangle$  iff  $M \in \text{Tran}(x)$ . For the other translation, we compute

$$\begin{aligned}
\langle\!\langle \Phi(x) \rangle\!\rangle &= \langle\!\langle \bigvee_{M \in \text{Tran}(x)} \left( \bigwedge_{(a,t) \in M} \langle a \rangle t \wedge \bigwedge_{(b,u) \notin M} \neg \langle b \rangle u \right) \rangle\!\rangle \\
&= \bigcup_{M \in \text{Tran}(x)} \left( \bigcap_{(a,t) \in M} \{M' \mid (a,t) \in M'\} \cap \bigcap_{(b,u) \notin M} \{M' \mid (b,u) \notin M'\} \right) \\
&= \bigcup_{M \in \text{Tran}(x)} \left( \{M' \mid \forall (a,t) \in M : (a,t) \in M'\} \right. \\
&\quad \left. \cap \{M' \mid \forall (b,u) \notin M : (b,u) \notin M'\} \right) \\
&= \bigcup_{M \in \text{Tran}(x)} (\{M' \mid M \subseteq M'\} \cap \{M' \mid M' \subseteq M\}) \\
&= \bigcup_{M \in \text{Tran}(x)} M = \text{Tran}(x).
\end{aligned}$$

$\mathcal{D}_1 \leq_m \mathcal{D}_2$  implies  $da(\mathcal{D}_1) \leq_m da(\mathcal{D}_2)$ :

Let  $\mathcal{D}_1 = (S_1, S_1^0, \dashrightarrow_1, \longrightarrow_1)$ ,  $\mathcal{D}_2 = (S_2, S_2^0, \dashrightarrow_2, \longrightarrow_2)$  be DMTS and assume  $\mathcal{D}_1 \leq_m \mathcal{D}_2$ . Then we have a modal refinement relation (in the DMTS sense)  $R \subseteq S_1 \times S_2$ . Now let  $(s_1, s_2) \in R$  and  $M_1 \in \text{Tran}_1(s_1)$ , and define

$$M_2 = \{(a, t_2) \mid s_2 \dashrightarrow_2^a t_2, \exists(a, t_1) \in M_1 : (t_1, t_2) \in R\}.$$

The condition

$$\forall(a, t_2) \in M_2 : \exists(a, t_1) \in M_1 : (t_1, t_2) \in R$$

in the definition of AA refinement is satisfied by construction. For the inverse condition, let  $(a, t_1) \in M_1$ , then  $s_1 \dashrightarrow_1^a t_1$ , so by DMTS refinement, there is  $t_2 \in S_2$  with  $s_2 \dashrightarrow_2^a t_2$  and  $(t_1, t_2) \in R$ , whence  $(a, t_2) \in M_2$  by construction.

We are left with showing that  $M_2 \in \text{Tran}_2(s_2)$ . First we notice that by construction, indeed  $s_2 \dashrightarrow_2^a t_2$  for all  $(a, t_2) \in M_2$ . Now let  $s_2 \longrightarrow N_2$ ; we need to show that  $N_2 \cap M_2 \neq \emptyset$ .

By DMTS refinement, we have  $s_1 \longrightarrow N_1$  such that  $\forall(a, t_1) \in N_1 : \exists(a, t_2) \in N_2 : (t_1, t_2) \in R$ . We know that  $N_1 \cap M_1 \neq \emptyset$ , so let  $(a, t_1) \in N_1 \cap M_1$ . Then there also is  $(a, t_2) \in N_2$  with  $(t_1, t_2) \in R$ . But  $(a, t_2) \in N_2$  implies  $s_2 \dashrightarrow_2^a t_2$ , hence  $(a, t_2) \in M_2$ .

$da(\mathcal{D}_1) \leq_m da(\mathcal{D}_2)$  implies  $\mathcal{D}_1 \leq_m \mathcal{D}_2$ :

Let  $\mathcal{D}_1 = (S_1, S_1^0, \dashrightarrow_1, \longrightarrow_1)$ ,  $\mathcal{D}_2 = (S_2, S_2^0, \dashrightarrow_2, \longrightarrow_2)$  be DMTS and assume  $da(\mathcal{D}_1) \leq_m da(\mathcal{D}_2)$ . Then we have a modal refinement relation (in the AA sense)  $R \subseteq S_1 \times S_2$ . Let  $(s_1, s_2) \in R$ .

Let  $s_1 \dashrightarrow_1^a t_1$ , then we cannot have  $s_1 \longrightarrow \emptyset$ . Let  $M_1 = \{(a, t_1)\} \cup \bigcup\{N_1 \mid s_1 \longrightarrow N_1\}$ , then  $M_1 \in \text{Tran}_1(s_1)$  by construction. This implies that there is  $M_2 \in \text{Tran}_2(s_2)$  and  $(a, t_2) \in M_2$  with  $(t_1, t_2) \in R$ , but then also  $s_2 \dashrightarrow_2^a t_2$  as was to be shown.

Let  $s_2 \longrightarrow N_2$  and assume, for the sake of contradiction, that there is no  $s_1 \longrightarrow N_1$  for which  $\forall(a, t_1) \in N_1 : \exists(a, t_2) \in N_2 : (t_1, t_2) \in R$  holds. Then for each  $s_1 \longrightarrow N_1$ , there is an element  $(a_{N_1}, t_{N_1}) \in N_1$  for which there is no  $(a_{N_1}, t_2) \in N_2$  with  $(t_{N_1}, t_2) \in R$ .

Let  $M_1 = \{(a_{N_1}, t_{N_1}) \mid s_1 \longrightarrow N_1\}$ , then  $M_1 \in \text{Tran}_1(s_1)$  by construction. Hence we have  $M_2 \in \text{Tran}_2(s_2)$  satisfying the conditions in the definition of AA refinement. By construction of  $\text{Tran}_2(s_2)$ ,  $N_2 \cap M_2 \neq \emptyset$ , so let  $(a, t_2) \in N_2 \cap M_2$ . Then there exists  $(a, t_1) \in M_1$  for which  $(t_1, t_2) \in R$ , in contradiction to the definition of  $M_1$ .

$\mathcal{A}_1 \leq_m \mathcal{A}_2$  implies  $ad(\mathcal{A}_1) \leq_m ad(\mathcal{A}_2)$ :

Let  $\mathcal{A}_1 = (S_1, S_1^0, \text{Tran}_1)$ ,  $\mathcal{A}_2 = (S_2, S_2^0, \text{Tran}_2)$  be AA, with DMTS translations  $(D_1, D_1^0, \longrightarrow_1, \dashrightarrow_1)$ ,  $(D_2, D_2^0, \longrightarrow_2, \dashrightarrow_2)$ , and assume  $\mathcal{A}_1 \leq_m \mathcal{A}_2$ . Then we have a modal refinement relation (in the AA sense)  $R \subseteq S_1 \times S_2$ . Define

$R' \subseteq D_1 \times D_2$  by

$$\begin{aligned} R' = \{ (M_1, M_2) \mid \exists (s_1, s_2) \in R : M_1 \in \text{Tran}_1(s_1), M_2 \in \text{Tran}(s_2), \\ \forall (a, t_1) \in M_1 : \exists (a, t_2) \in M_2 : (t_1, t_2) \in R, \\ \forall (a, t_2) \in M_2 : \exists (a, t_1) \in M_1 : (t_1, t_2) \in R \}. \end{aligned}$$

We show that  $R'$  is a modal refinement in the DMTS sense. Let  $(M_1, M_2) \in R'$ .

Let  $M_2 \xrightarrow{a} N_2$ . By construction of  $\xrightarrow{a}$ , there is  $(a, t_2) \in M_2$  such that  $N_2 = \{(a, M'_2) \mid M'_2 \in \text{Tran}_2(t_2)\}$ . Then  $(M_1, M_2) \in R'$  implies that there must be  $(a, t_1) \in M_1$  for which  $(t_1, t_2) \in R$ , and we can define  $N_1 = \{(a, M'_1) \mid M'_1 \in \text{Tran}_1(t_1)\}$ , whence  $M_1 \xrightarrow{a} N_1$ .

We show that  $\forall (a, M'_1) \in N_1 : \exists (a, M'_2) \in N_2 : (M'_1, M'_2) \in R'$ : Let  $(a, M'_1) \in N_1$ , then  $M'_1 \in \text{Tran}_1(t_1)$ . From  $(t_1, t_2) \in R$  we hence get  $M'_2 \in \text{Tran}_2(t_2)$ , and then  $(a, M'_2) \in N_2$  by construction of  $N_2$  and  $(M'_1, M'_2) \in R'$  due to the conditions of AA refinement (applied to  $(t_1, t_2) \in R$ ).

Let  $M_1 \xrightarrow{a} M'_1$ , then we have  $M_1 \xrightarrow{a} N_1$  for which  $(a, M'_1) \in N_1$  by construction of  $\xrightarrow{a}$ . This in turn implies that there must be  $(a, t_1) \in M_1$  such that  $N_1 = \{(a, M'_1) \mid M'_1 \in \text{Tran}_1(t_1)\}$ , and then by  $(M_1, M_2) \in R'$ , we get  $(a, t_2) \in M_2$  for which  $(t_1, t_2) \in R$ . Let  $N_2 = \{(a, M'_2) \mid M'_2 \in \text{Tran}_2(t_2)\}$ , then  $M_2 \xrightarrow{a} N_2$  and hence  $M_2 \xrightarrow{a} M'_2$  for all  $(a, M'_2) \in N_2$ . On the other hand, the argument in the previous paragraph shows that there is  $(a, M'_2) \in N_2$  for which  $(M'_1, M'_2) \in R'$ .

We miss to show that  $R'$  is initialized. Let  $M_1^0 \in D_1^0$ , then we have  $s_1^0 \in S_1^0$  with  $M_1^0 \in \text{Tran}_1(s_1^0)$ . As  $R$  is initialized, this entails that there is  $s_2^0 \in S_2^0$  with  $(s_1^0, s_2^0) \in R$ , which gives us  $M_2^0 \in \text{Tran}_2(s_2^0)$  which satisfies the AA refinement conditions, whence  $(M_1^0, M_2^0) \in R'$ .

$ad(\mathcal{A}_1) \leq_m ad(\mathcal{A}_2)$  implies  $\mathcal{A}_1 \leq_m \mathcal{A}_2$ :

Let  $\mathcal{A}_1 = (S_1, S_1^0, \text{Tran}_1)$ ,  $\mathcal{A}_2 = (S_2, S_2^0, \text{Tran}_2)$  be AA, with DMTS translations  $(D_1, D_1^0, \xrightarrow{a}_1, \xrightarrow{a}_1)$ ,  $(D_2, D_2^0, \xrightarrow{a}_2, \xrightarrow{a}_2)$ , and assume  $ad(\mathcal{A}_1) \leq_m ad(\mathcal{A}_2)$ . Then we have a modal refinement relation (in the DMTS sense)  $R \subseteq D_1 \times D_2$ . Define  $R' \subseteq S_1 \times S_2$  by

$$R' = \{(s_1, s_2) \mid \forall M_1 \in \text{Tran}_1(s_1) : \exists M_2 \in \text{Tran}_2(s_2) : (M_1, M_2) \in R\};$$

we will show that  $R'$  is an AA modal refinement.

Let  $(s_1, s_2) \in R'$  and  $M_1 \in \text{Tran}_1(s_1)$ , then by construction of  $R'$ , we have  $M_2 \in \text{Tran}_2(s_2)$  with  $(M_1, M_2) \in R$ .

Let  $(a, t_2) \in M_2$  and define  $N_2 = \{(a, M'_2) \mid M'_2 \in \text{Tran}_2(t_2)\}$ , then  $M_2 \xrightarrow{a} N_2$ . Now  $(M_1, M_2) \in R$  implies that there must be  $M_1 \xrightarrow{a} N_1$  satisfying  $\forall (a, M'_1) \in N_1 : \exists (a, M'_2) \in N_2 : (M'_1, M'_2) \in R$ . We have  $(a, t_1) \in M_1$  such that  $N_1 = \{(a, M'_1) \mid M'_1 \in \text{Tran}_1(t_1)\}$ ; we only miss to show that  $(t_1, t_2) \in R'$ . Let  $M'_1 \in \text{Tran}_1(t_1)$ , then  $(a, M'_1) \in N_1$ , hence there is  $(a, M'_2) \in N_2$  with  $(M'_1, M'_2) \in R$ , but  $(a, M'_2) \in N_2$  also entails  $M'_2 \in \text{Tran}_2(t_2)$ .

Let  $(a, t_1) \in M_1$  and define  $N_1 = \{(a, M'_1) \mid M'_1 \in \text{Tran}_1(t_1)\}$ , then  $M_1 \xrightarrow{a} N_1$ . Now let  $(a, M'_1) \in N_1$ , then  $M_1 \xrightarrow{a} M'_1$ , hence we have  $M_2 \xrightarrow{a} M'_2$  for

some  $(M'_1, M'_2) \in R$  by modal refinement. By construction of  $\dashv\dashv$ , this implies that there is  $M_2 \dashv\dashv_2 N_2$  with  $(a, M_2) \in N_2$ , and we have  $(a, t_2) \in M_2$  for which  $N_2 = \{(a, M''_2) \mid M''_2 \in \text{Tran}_2(t_2)\}$ . Now if  $M''_1 \in \text{Tran}_1(t_1)$ , then  $(a, M''_1) \in N_1$ , hence there is  $(a, M''_2) \in N_2$  with  $(M''_1, M''_2) \in R$ , but  $(a, M''_2) \in N_2$  also gives  $M''_2 \in \text{Tran}_2(t_2)$ .

We miss to show that  $R'$  is initialized. Let  $s_1^0 \in S_1^0$ , then  $\text{Tran}_1(s_1^0) \neq \emptyset$ , hence there is  $M_1^0 \in \text{Tran}_1(s_1^0)$ . As  $R$  is initialized, this gets us  $M_2^0 \in D_2$  with  $(M_1^0, M_2^0) \in R$ , but  $M_2^0 \in \text{Tran}_2(s_2^0)$  for some  $s_2^0 \in S_2^0$ , and then  $(s_1^0, s_2^0) \in R'$ .  $\square$

*Proof (of Lemma 8).* It is shown in [11] that any Hennessy-Milner formula is equivalent to one in so-called *strong normal form*, i.e. of the form  $\bigvee_{i \in I} (\bigwedge_{j \in J_i} \langle a_{ij} \rangle \phi_{ij} \wedge \bigwedge_{a \in \Sigma} [a] \psi_{i,a})$  for HML formulas  $\phi_{ij}, \psi_{i,a}$  which are also in strong normal form. Now we can replace the  $\phi_{ij}, \psi_{i,a}$  by (new) variables  $x_{ij}, y_{i,a}$  and add declarations  $\Delta_2(x_{ij}) = \phi_{ij}, \Delta_2(y_{i,a}) = \psi_{i,a}$  to arrive at an expression in which all formulae are of the form  $\Delta_2(x) = \bigvee_{i \in I} (\bigwedge_{j \in J_i} \langle a_{ij} \rangle x_{ij} \wedge \bigwedge_{a \in \Sigma} [a] x_{i,a})$ .

Now for each such formula, replace (recursively)  $x$  by new variables  $\{\tilde{x}^i \mid i \in I\}$  and set  $\Delta_2(\tilde{x}^i) = \bigwedge_{j \in J_i} \langle a_{ij} \rangle (\bigvee_k \tilde{x}_{ij}^k) \wedge \bigwedge_{a \in \Sigma} [a] (\bigvee_k \tilde{x}_{i,a}^k)$ . Using initial variables  $X_2^0 = \{\tilde{x}^i \mid x \in X_1^0\}$ , the so-constructed  $\nu$ -calculus expression is equivalent to the original one.  $\square$

*Proof (of Theorem 17).* This follows directly from the fact, easy to prove because of the purely syntactic translation, that  $hd(\mathcal{N}_1 \vee \mathcal{N}_2) = hd(\mathcal{N}_1) \vee hd(\mathcal{N}_2)$  and similarly for conjunction, where the operations on the right-hand side are the ones defined for DMTS in [6]. Given this, the theorem follows from similar properties for the DMTS operations [6].  $\square$

*Proof (of Theorem 18).* Only distributivity remains to be verified. Let  $\mathcal{N}_i = (X_i, X_i^0, \Delta_i)$ , for  $i = 1, 2, 3$ , be  $\nu$ -calculus expressions in normal form. The set of variables of both  $\mathcal{N}_1 \wedge (\mathcal{N}_2 \vee \mathcal{N}_3)$  and  $(\mathcal{N}_1 \wedge \mathcal{N}_2) \vee (\mathcal{N}_1 \wedge \mathcal{N}_3)$  is  $X_1 \times (X_2 \times X_3)$ , and one easily sees that the identity relation is a two-sided modal refinement. Things are similar for the other distributive law.  $\square$

*Proof (of Theorem 22).* We have already seen that the class of  $\nu$ -calculus expressions forms a lattice, up to  $\equiv_m$ , under  $\wedge$  and  $\vee$ , and by Theorem 21,  $/$  is the residual, up to  $\equiv_m$ , of  $\parallel$ . We only miss to show that  $\mathbf{U}$  is indeed the unit of  $\parallel$ ; all other properties (such as distributivity of  $\parallel$  over  $\vee$  or  $\mathcal{N} \parallel \perp \equiv_m \perp$ ) follow.

We show that  $\mathcal{A} \parallel_{\mathbf{A}} \mathbf{U} \equiv_m \mathcal{A}$  for all AA  $\mathcal{A}$ ; the analogous property for  $\nu$ -calculus expressions follows from the translations. Let  $\mathcal{A} = (S, S^0, \text{Tran})$  be an AA and define  $R = \{((s, u), s) \mid s \in S\}$ ; we show that  $R$  is a two-sided modal refinement. Let  $((s, u), s) \in R$  and  $M \in \text{Tran}(s, u)$ , then there must be  $M_1 \in \text{Tran}(s)$  for which  $M = M_1 \parallel (\Sigma \times \{u\})$ . Thus  $M_1 = \{(a, t) \mid (a, (t, u)) \in M\}$ . Then any element of  $M$  has a corresponding one in  $M_1$ , and vice versa, and their states are related by  $R$ . For the other direction, let  $M_1 \in \text{Tran}(s)$ , then  $M = M_1 \parallel (\Sigma \times \{u\}) = \{(a, (t, u)) \mid (a, t) \in M_1\} \in \text{Tran}(s, u)$ , and the same argument applies.  $\square$